

# Affine Circle Geometry over Quaternion Skew Fields

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## Abstract

We investigate the affine circle geometry arising from a quaternion skew field and one of its maximal commutative subfields.

## 1 Introduction

### 1.1

The present paper is concerned with the chain geometry  $\Sigma(K, L)$  (cf. [1]) on a field extension  $L/K$ , where  $K$  is a maximal commutative subfield of a quaternion skew field  $L$ . Thus  $L$  is not a  $K$ -algebra. This has many geometric consequences. Best known is probably that three distinct points do not determine a unique chain. As in ordinary Möbius-geometry, it is possible to obtain an affine plane by deleting one point, but a more sophisticated technique is necessary in order to define the lines of this plane. We take a closer look on this construction from two different points of view, starting either from a spread of lines associated to  $\Sigma(K, L)$  or the point model of this spread on the Klein quadric. The chains of  $\Sigma(K, L)$  yield the lines, degenerate circles and non-degenerate circles of such an affine plane. We establish some properties of these circles and show that degenerate circles are affine Baer subplanes. If  $K$  is Galois over the centre of  $L$  then each non-degenerate circle can be written as intersection of two affine Hermitian varieties.

We encourage the reader to compare our results with the survey article [9] on chain geometry over an algebra and [10]. There is an extensive literature on the real quaternions. A lot of references can be found, e.g., in [1], [2], [5], [15], [16].

### 1.2

Throughout this paper  $L$  will denote a quaternion skew field with centre  $Z$  and  $K$  will be a maximal commutative subfield of  $L$ . The following exposition follows [4], [8], [13, pp.168–171].

Choose any element  $a \in K \setminus Z$  with minimal equation, say

$$a^2 + a\lambda_1 + \mu_1 = 0 \quad (\lambda_1, \mu_1 \in Z).$$

If  $K/Z$  is Galois then

$$(\bar{\phantom{x}}) : K \rightarrow K, \quad u = \xi + a\eta \mapsto \bar{u} := \xi - (\lambda_1 + a)\eta \quad (\xi, \eta \in Z)$$

is an automorphism of order 2 fixing  $Z$  elementwise<sup>1</sup>. There exists an element  $i \in L \setminus K$  such that

$$i^{-1}ui = \bar{u} \quad \text{for all } u \in K,$$

whence

$$ui = i\bar{u} \quad \text{for all } u \in K. \quad (1)$$

If  $K/Z$  is not Galois then, obviously,  $\text{Char}K = 2$  and  $\lambda_1 = 0$ . The mapping

$$D : K \rightarrow K, \quad u = \xi + a\eta \mapsto u^D := a\eta \quad (\xi, \eta \in Z)$$

is additive and satisfies  $(uu')^D = u^Du' + uu'^D$  for all  $u, u' \in K$ , i.e.,  $D$  is a derivation of  $K$ . There exists an  $i \in L \setminus K$  such that

$$a^{-1}ia = i + 1$$

which leads to the rule

$$ui = iu + u^D \quad \text{for all } u \in K. \quad (2)$$

In every case the element  $i$  has a minimal equation over  $Z$ , say

$$i^2 + i\lambda_2 + \mu_2 = 0 \quad (\lambda_2, \mu_2 \in Z).$$

If  $K/Z$  is Galois then  $i^2 \in Z$ , whence  $\lambda_2 = 0$ . If  $K/Z$  is not Galois then  $i$  and  $i + 1$  have the same minimal equation. This implies  $\lambda_2 = 1$ . The mapping

$$A : L \rightarrow L, \quad u + iv \mapsto \begin{cases} \bar{u} - iv & : K/Z \text{ Galois,} \\ u + v + vi & : K/Z \text{ not Galois,} \end{cases} \quad (u, v \in K) \quad (3)$$

is an involutory antiautomorphism of  $L$  fixing  $K$ . The norm of  $x \in L$  is given by  $N(x) := x^Ax$ .

### 1.3

The mappings  $(\bar{\phantom{x}})$  and  $D$  allow, respectively, the following geometric interpretations:

Let  $\mathbf{V}$  be a right vector space over  $Z$ ,  $\dim \mathbf{V} \geq 2$ . We are extending  $\mathbf{V}$  to  $\mathbf{V} \otimes_Z K$  with  $\mathbf{v} \in \mathbf{V}$  to be identified with  $\mathbf{v} \otimes 1$ . Then define a mapping  $\mathbf{V} \otimes_Z K \rightarrow \mathbf{V} \otimes_Z K$  by

$$\sum_{\mathbf{v} \in \mathbf{V}} \mathbf{v} \otimes k_{\mathbf{v}} \mapsto \begin{cases} \sum_{\mathbf{v} \in \mathbf{V}} \mathbf{v} \otimes \bar{k}_{\mathbf{v}} & : K/Z \text{ Galois,} \\ \sum_{\mathbf{v} \in \mathbf{V}} \mathbf{v} \otimes k_{\mathbf{v}}^D & : K/Z \text{ not Galois,} \end{cases} \quad (k_{\mathbf{v}} \in K).$$

By abuse of notation, this mapping will also be written as  $(\bar{\phantom{x}})$  and  $D$ , respectively.

In terms of the projective spaces  $\mathcal{P}_Z(\mathbf{V})$  and  $\mathcal{P}_K(\mathbf{V} \otimes_Z K)$  the first projective space is being embedded in the second one as a Baer subspace. If  $\mathbf{x}K$  is a point of  $\mathcal{P}_K(\mathbf{V} \otimes_Z K) \setminus \mathcal{P}_Z(\mathbf{V})$  then through this point there is a unique line of  $\mathcal{P}_K(\mathbf{V} \otimes_Z K)$  containing more than one point of  $\mathcal{P}_Z(\mathbf{V})$ . That line is given by

$$\mathbf{x}K \vee \bar{\mathbf{x}}K \quad \text{and} \quad \mathbf{x}K \vee (\mathbf{x}^D)K,$$

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<sup>1</sup>By an appropriate choice of  $a$  it would be possible to have  $\lambda_1 = 0$  ( $\text{Char}K \neq 2$ ) or  $\lambda_1 = 1$  ( $\text{Char}K = 2$ ).

respectively<sup>2</sup>. Note that defining a mapping by setting  $\mathbf{x}K \mapsto (\mathbf{x}^D)K$  is ambiguous, since

$$(\mathbf{x}u)^D = \mathbf{x}^D u + \mathbf{x}u^D \quad \text{for all } \mathbf{x} \in \mathbf{V} \otimes_Z K, u \in K.$$

We give a second interpretation in terms of affine planes<sup>3</sup>:

**Lemma 1** *Let  $\mathbf{W}$  be a right vector space over  $K$ ,  $\dim \mathbf{W} = 2$ , and let  $\{\mathbf{u}, \mathbf{v}\}$  be a basis of  $\mathbf{W}$ . Then*

$$\begin{aligned} \{\mathbf{u}k + \mathbf{v}\bar{k} \mid k \in K\} &: K/Z \text{ Galois,} \\ \{\mathbf{u}k + \mathbf{v}k^D \mid k \in K\} &: K/Z \text{ not Galois,} \end{aligned} \quad (4)$$

*is an affine Baer subplane (over  $Z$ ) of the affine plane on  $\mathbf{W}$ .*

*Proof.* If  $\mathbf{u}'$  and  $\mathbf{v}'$  are linearly independent vectors of  $\mathbf{W}$  then the set of all linear combinations of  $\mathbf{u}'$  and  $\mathbf{v}'$  with coefficients in  $Z$  is an affine Baer subplane over  $Z$ . Write  $k = \xi + a\eta$  with  $\xi, \eta \in Z$ .

If  $K/Z$  is Galois then

$$\mathbf{u}k + \mathbf{v}\bar{k} = (\mathbf{u} + \mathbf{v})\xi + (\mathbf{u}a - \mathbf{v}(\lambda_1 + a))\eta.$$

The vectors  $\mathbf{u} + \mathbf{v}$  and  $\mathbf{u}a - \mathbf{v}(\lambda_1 + a)$  are linearly independent, since otherwise we would have the contradiction  $\bar{a} = -\lambda_1 - a = a$ .

If  $K/Z$  is not Galois then

$$\mathbf{u}k + \mathbf{v}k^D = \mathbf{u}\xi + (\mathbf{u} + \mathbf{v})a\eta.$$

The vectors  $\mathbf{u}$  and  $(\mathbf{u} + \mathbf{v})a$  are linearly independent. ■

## 2 Projective Chain Geometry on $L/K$

### 2.1

Let  $L/K$  be given as before. Following [1, p.320ff.] we obtain an incidence structure  $\Sigma(K, L)$  as follows: The points of  $\Sigma(K, L)$  are the points of the projective line over  $L$ , viz.  $\mathcal{P}_L(L^2)$ , the blocks, now called chains, are the  $K$ -sublines of  $\mathcal{P}_L(L^2)$ . However, in contrast to [1], we shall regard  $L^2$  as right vector space over  $K$  rather than  $L$ . Each  $s$ -dimensional subspace of  $L^2$  (over  $L$ ) is  $2s$ -dimensional over  $K$ , whence  $\mathcal{P}_K(L^2) =: \mathcal{P}_K$  is 3-dimensional. The points of  $\Sigma(K, L)$  now appear as lines of a spread of  $\mathcal{P}_K$ , say  $\mathcal{S}_{L/K}$ ; cf. [6], [7]. If  $t$  is a line of  $\mathcal{P}_K$  not contained in  $\mathcal{S}_{L/K}$  then through each point of  $t$  there goes exactly one line of  $\mathcal{S}_{L/K}$ . The subset  $\mathcal{C}$  of  $\mathcal{S}_{L/K}$  arising in this way is a chain of  $\Sigma(K, L)$ . We call  $t$  a transversal line of the chain  $\mathcal{C}$ . If  $L/K$  is not Galois then each chain has exactly one transversal line, otherwise exactly two transversal lines that are interchanged under the non-projective collineation

$$\iota : \mathcal{P}_K \rightarrow \mathcal{P}_K, \quad (l_0, l_1)K \mapsto (l_0i, l_1i)K. \quad (5)$$

Cf. [8, Theorem 2], [11].

<sup>2</sup>At least in the first case this is very well known.

<sup>3</sup>Cf. the concept of ‘Minimalkoordinaten’ described, e.g., in [17, p.35]

## 2.2

Write  $\mathcal{L}$  for the set of lines of  $\mathcal{P}_K$  and  $\gamma : \mathcal{L} \rightarrow \widehat{\mathcal{P}}_K$  for the Klein mapping. Here  $\widehat{\mathcal{P}}_K$  is the ambient space of the Klein quadric  $\mathcal{Q} := \mathcal{L}^\gamma$ . The underlying vector space of  $\widehat{\mathcal{P}}_K$  is  $L^2 \wedge L^2$  (over  $K$ ). In [8, Theorem 1] it is shown that there is a unique 5-dimensional Baer subspace  $\Pi_Z$  (over  $Z$ ) of  $\widehat{\mathcal{P}}_K$  such that

$$\mathcal{S}_{L/K}^\gamma = \Pi_Z \cap \mathcal{Q}.$$

With respect to  $\Pi_Z$  the set  $\mathcal{S}_{L/K}^\gamma$  is an oval quadric, i.e. a quadric without lines. A subset  $\mathcal{C}$  of  $\mathcal{S}_{L/K}$  is a chain if, and only if, there exists a 3-dimensional subspace  $\mathcal{X}$  of  $\widehat{\mathcal{P}}_K$  such that<sup>4</sup>

$$\mathcal{X} \cap \Pi_Z \text{ is a 3-dimensional subspace of } \Pi_Z, \quad (6)$$

$$\mathcal{C}^\gamma = \mathcal{X} \cap \mathcal{S}_{L/K}^\gamma \text{ is an elliptic quadric of } \mathcal{X} \cap \Pi_Z \text{ (over } Z), \quad (7)$$

$$\mathcal{X} \cap \mathcal{Q} \text{ contains a line of } \widehat{\mathcal{P}}_K; \quad (8)$$

cf. [8, Theorem 1].

## 2.3

The automorphism group of  $\Sigma(K, L)$  is formed by all bijections of  $\mathcal{S}_{L/K}$  taking chains to chains in both directions. If  $\kappa$  is a collineation or a duality of  $\mathcal{P}_K$  with  $\mathcal{S}_{L/K}^\kappa = \mathcal{S}_{L/K}$  then  $\kappa$  is yielding an automorphism of  $\Sigma(K, L)$ . Conversely, according to [12] and [8, Theorem 4], each automorphism of  $\Sigma(K, L)$  can be induced by an automorphic collineation or duality of  $\mathcal{S}_{L/K}$ , say  $\kappa$ . This  $\kappa$  is uniquely determined for  $K/Z$  not being Galois, otherwise the product of  $\iota$  (cf. formula (5)) and  $\kappa$  is the only other solution.

Transferring these results to  $\widehat{\mathcal{P}}_K$  establishes that an automorphic collineation  $\mu$  of the Klein quadric is the  $\gamma$ -transform of an automorphic collineation or duality of  $\mathcal{S}_{L/K}$  if, and only if,  $\Pi_Z$  is invariant under  $\mu$ . If  $K/Z$  is Galois, then the  $\gamma$ -transform of the collineation  $\iota$  (cf. (5)) is the Baer involution of  $\widehat{\mathcal{P}}_K$  fixing  $\Pi_Z$  pointwise. See [8, Theorem 4].

## 2.4

Let  $\mathcal{C}_0$  and  $\mathcal{C}_1$  be two chains with a common element, say  $p \in \mathcal{S}_{L/K}$ . We say that  $\mathcal{C}_0$  is **tangent** to  $\mathcal{C}_1$  at  $p$  if there exist transversal lines  $t_i$  of  $\mathcal{C}_i$  ( $i = 0, 1$ ) such that  $p, t_0, t_1$  are in one pencil of lines. This is a reflexive and symmetric relation.

If  $K/Z$  is Galois then there is also an orthogonality relation on the set of chains: If  $\mathcal{C}_i$  ( $i = 0, 1$ ) are chains with transversal lines  $t_i, t_i^\iota$ , respectively, then  $\mathcal{C}_0$  is said to be **orthogonal** to  $\mathcal{C}_1$  if  $t_0$  intersects both  $t_1$  and  $t_1^\iota$ . This relation is symmetric, since  $\iota$  is an involution. Given two orthogonal chains their transversal lines form a skew quadrilateral.

The two definitions above are not given in an intrinsic way. However, both relations are invariant under automorphic collineations and dualities of  $\mathcal{S}_{L/K}$  and hence invariant under automorphisms of  $\Sigma(K, L)$ .

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<sup>4</sup>If  $L$  is the skew field of real quaternions then  $K$  is a field of complex numbers and  $Z$  the field of real numbers. Here conditions (6) and (7) are already sufficient to characterize the  $\gamma$ -images of chains.

The proofs of the following results are left to the reader: Chains  $\mathcal{C}_0, \mathcal{C}_1$  are tangent at  $p \in \mathcal{C}_0 \cap \mathcal{C}_1$  if, and only if, their images under the Klein mapping are quadrics with the same tangent plane at the point  $p^\gamma$ . A chain  $\mathcal{C}_0$  is orthogonal to a chain  $\mathcal{C}_1$  if, and only if, the subspace of  $\widehat{\mathcal{P}}_K$  spanned by  $\mathcal{C}_0^\gamma$  contains the orthogonal subspace (with respect to the Klein quadric) of  $\mathcal{C}_1^\gamma$ .

### 3 Affine Circle Geometry on $L/K$

#### 3.1

With the notations introduced in section 2, select one line of  $\mathcal{S}_{L/K}$  and label it  $\infty$ . Let  $\tilde{\mathcal{A}}$  be a (projective) plane of  $\mathcal{P}_K$  through  $\infty$  and write  $\mathcal{A} := \tilde{\mathcal{A}} \setminus \infty$ . Then  $\mathcal{A}$  can be viewed as an affine plane with  $\infty$  as line at infinity. The mapping

$$\rho : \mathcal{S}_{L/K} \setminus \{\infty\} \rightarrow \mathcal{A}, \quad s \mapsto \mathcal{A} \cap s \quad (9)$$

is well-defined and bijective. A chain  $\mathcal{C}$  containing  $\infty$  yields an affine line  $(\mathcal{C} \setminus \{\infty\})^\rho$  if, and only if,  $\mathcal{C}$  has a transversal line in  $\tilde{\mathcal{A}}$ . Two chains with transversal lines in  $\tilde{\mathcal{A}}$  yield parallel lines if, and only if, the chains are tangent at  $\infty$ .

If  $\tilde{\mathcal{A}}'$  is any plane through  $\infty$  then, with  $\mathcal{A}' := \tilde{\mathcal{A}}' \setminus \infty$ , the mapping

$$\beta : \mathcal{A} \rightarrow \mathcal{A}', \quad \mathcal{A} \cap s \mapsto \mathcal{A}' \cap s \quad (s \in \mathcal{S}_{L/K} \setminus \{\infty\})$$

is a well-defined bijection<sup>5</sup>. This  $\beta$  is an affinity if either  $\tilde{\mathcal{A}}' = \tilde{\mathcal{A}}$  or  $\tilde{\mathcal{A}}' = \tilde{\mathcal{A}}^\iota$  [6, Theorem 5]; the second alternative is only possible when  $K/Z$  is Galois.

#### 3.2

The group of automorphic collineations of  $\mathcal{S}_{L/K}$  operates 3-fold transitively on the lines of  $\mathcal{S}_{L/K}$  [1, p.322]. Thus we may transfer  $\infty$  to the line given by  $(0, 1)L$ . Moreover, for all  $c \in L, c \neq 0$

$$(l_0, l_1)K \mapsto (cl_0, cl_1)K \quad ((0, 0) \neq (l_0, l_1) \in L^2)$$

is an automorphic collineation of  $\mathcal{S}_{L/K}$  fixing  $\infty$ . Hence, without loss of generality, we may assume in the sequel that

$$\infty = \mathcal{P}_K((0, 1)L) \text{ and } \tilde{\mathcal{A}} = (1, 0)K \vee \infty.$$

Then the mapping (9) becomes

$$\mathcal{P}_K((l_0, l_1)L) \mapsto (1, l_1 l_0^{-1})K. \quad (10)$$

We shall identify  $\mathcal{A}$  with  $L$  via<sup>6</sup>  $(1, l)K \equiv l$ . Thus  $L$  gets the structure of an affine plane over  $K$ . We shall emphasize this by writing  $\mathcal{A}_K(L)$  rather than  $L$ .

<sup>5</sup>One could also select some point  $A \in \infty$  and then obtain an affine plane by a dual construction.

<sup>6</sup>This is accordance with the inhomogeneous notation used in [1].

**Theorem 1** *Let  $\kappa$  be an automorphic collineation or duality of  $\mathcal{S}_{L/K}$  fixing  $\infty$ . Then there exist elements  $m_0, m_1, m \in L$ ,  $m_0, m_1 \neq 0$  and an automorphism or antiautomorphism  $J$  of  $L$  with  $K^J = K$  such that*

$$x^{\rho^{-1}\kappa\rho} = m_1 x^J m_0 + m \quad \text{for all } x \in L. \quad (11)$$

*The additional conditions*

$$J \text{ is an automorphism of } L, \quad (12)$$

$$m_0 \in K \text{ or, only if } K/Z \text{ is Galois, } m_0 i^{-1} \in K \quad (13)$$

*together are necessary and sufficient for  $\rho^{-1}\kappa\rho$  to be an affinity of  $\mathcal{A}_K(L)$ .*

*Proof.* The assertion in formula (11) is obviously true.

Now suppose that  $\rho^{-1}\kappa\rho$  is an affinity of  $\mathcal{A}_K(L)$ . Then  $\kappa$  has to take each chain with a transversal line in  $\tilde{\mathcal{A}}$  to a chain with a transversal line in  $\tilde{\mathcal{A}}$ . Hence  $\tilde{\mathcal{A}}^\kappa = \tilde{\mathcal{A}}$  or, only if  $K/Z$  is Galois,  $\tilde{\mathcal{A}}^\kappa = \tilde{\mathcal{A}}^i$ . Therefore  $\kappa$  cannot be a duality, so that  $J$  cannot be an antiautomorphism [8, Theorem 4]. Consequently,  $g : x \mapsto m_1 x^J m_0$  has to be a semilinear mapping of the right vector space  $L$  over  $K$ . We infer from

$$xk \xrightarrow{g} (m_1 x^J m_0)(m_0^{-1} k^J m_0) \quad \text{for all } x \in L, k \in K$$

that  $m_0^{-1} K m_0 = K$ . There are two possibilities: If

$$m_0^{-1} k m_0 = k \quad \text{for all } k \in K$$

then  $m_0$  is a non-zero element of  $K$ , since  $K$  is a maximal commutative subfield of  $L$ . On the other hand, however only if  $K/Z$  is Galois, also

$$m_0^{-1} k m_0 = \bar{k} \quad \text{for all } k \in K$$

is possible. Now, again using that  $K$  is maximal commutative, it follows from (1) that  $m_0 i^{-1} \in K$ .

The proof of the converse is a straightforward calculation. ■

### 3.3

If  $\mathcal{C}$  is a chain such that  $(\mathcal{C} \setminus \{\infty\})^\rho$  is not a line of  $\mathcal{A}_K(L)$  then  $(\mathcal{C} \setminus \{\infty\})^\rho$  will be named a **circle**. There are two kinds of circles: If  $\infty \in \mathcal{C}$  then the circle is called **degenerate**, otherwise **non-degenerate**. The following Lemma shows that distinct chains cannot define the same circle. In addition it establishes that a circle cannot be degenerate and non-degenerate at the same time:

**Lemma 2** *Let  $\mathcal{C}_0$  and  $\mathcal{C}_1$  be two chains such that  $\mathcal{C}_0 \setminus \{\infty\} = \mathcal{C}_1 \setminus \{\infty\}$ . Then  $\mathcal{C}_0 = \mathcal{C}_1$ .*

*Proof.* According to (6), (7), (8) there exists a 3-dimensional subspace  $\mathcal{X}_0$  of  $\hat{\mathcal{P}}_K$  with

$$\mathcal{C}_0^\gamma = \mathcal{X}_0 \cap \Pi_Z \cap \mathcal{Q}.$$

Since  $\mathcal{C}_0^\gamma$  is an oval quadric of  $\mathcal{X}_0 \cap \Pi_Z$  and  $Z$  is infinite,  $(\mathcal{C}_0 \setminus \{\infty\})^\gamma$  is still spanning  $\mathcal{X}_0$ . Repeating this, mutatis mutandis, for  $\mathcal{C}_1$  gives  $\mathcal{X}_0 = \mathcal{X}_1$ , whence  $\mathcal{C}_0 = \mathcal{C}_1$ , as required. ■

### 3.4

By Lemma 2, we may unambiguously speak of a line being **tangent** to a circle at some point  $P \in \mathcal{A}_K(L)$  or of circles **touching** at  $P$  if they arise from chains that are tangent at  $P^{\rho^{-1}}$ .

A degenerate circle has no tangent lines. A point  $P$  of a non-degenerate circle is called **regular** if there exists a tangent line of that circle at  $P$ . If such a circle is given as  $\mathcal{C}^\rho$ ,  $\mathcal{C}$  a chain, then  $P \in \mathcal{C}^\rho$  is regular if, and only if,  $P$  (regarded as point of  $\mathcal{A}$ ) is incident with a transversal line of  $\mathcal{C}$ . Thus a non-degenerate circle has either one or two regular points.

### 3.5

If  $K/Z$  is Galois then call two lines, or a circle and a line, or two circles of  $\mathcal{A}_K(L)$  **orthogonal** if they arise from orthogonal chains.

By virtue of the collineation  $\iota$  (cf. formula (5)), a line  $lK + m$  ( $l, m \in L$ ,  $l \neq 0$ ) is orthogonal to all lines being parallel to  $liK$ .

We introduce a unitary scalar product  $*$  on the right vector space  $L$  over  $K$  by setting

$$(u + iv) * (u' + iv') := \bar{u}u' + \mu_2 \bar{v}v' \quad \text{for all } u, u', v, v' \in K. \quad (14)$$

This scalar product is describing the orthogonality relation on lines from above. Moreover,  $(u + iv) * (u + iv) = N(u + iv)$ , whence the norm is a Hermitian form<sup>7</sup> on  $L$  over  $K$ .

It is easily seen that there exists no line orthogonal to a degenerate circle. The join of the two regular points of a non-degenerate circle is the only line being orthogonal to that circle. It will be called the **midline** of the circle. The midline is orthogonal to both tangent lines.

All affinities described in Theorem 1 are preserving orthogonality.

### 3.6

Let  $\mathcal{C}$  be a chain such that  $\Delta := (\mathcal{C} \setminus \{\infty\})^\rho$  is a degenerate circle. Then either there are two points or there is one point on the line  $\infty$  incident with transversal lines of  $\mathcal{C}$ . We call these points at infinity of  $\mathcal{A}_K(L)$  the **absolute points** or the **absolute directions** of  $\Delta$ . This terminology will be motivated in 3.10.

The group  $\text{AGL}(1, L)$  of all transformations (11) with  $m_0 = 1$  operates sharply 2-fold transitively on  $\mathcal{A}_K(L)$ . Thus each degenerate circle can be transferred under  $\text{AGL}(1, L)$  to a degenerate circle through 0 and 1. Write

$$L^\circ := \begin{cases} L \setminus (K \cup Ki) : K/Z \text{ Galois,} \\ L^\circ := L \setminus K : K/Z \text{ not Galois.} \end{cases}$$

Then, by [1, p.329] and (13), the degenerate circles through 0 and 1 are exactly the sets

$$cKc^{-1} \quad \text{with } c \in L^\circ. \quad (15)$$

From now on assume that a degenerate circle  $\Delta$  is given by (15). Let  $\mathcal{C}$  be the chain with transversal line  $(c, 0)K \vee (0, c)K$ . Then  $\Delta = (\mathcal{C} \setminus \{\infty\})^\rho$ , whence

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<sup>7</sup>If  $K/Z$  is not Galois then the norm does not seem to be a quadratic or Hermitian form on  $L$  over  $K$ .

$cK$  is an absolute direction of  $\Delta$ . Each affinity of  $\text{AGL}(1, L)$  (cf. formula (11)) with  $m_1, m \in cKc^{-1}$  ( $m_1 \neq 0$ ,  $m_0 = 1$  as before) takes  $\Delta$  onto  $\Delta$ .

**Theorem 2** *Each degenerate circle of  $\mathcal{A}_K(L)$  is an affine Baer subplane of  $\mathcal{A}_K(L)$  with the centre of  $L$  as underlying field.*

*Proof.* It is sufficient to show this for a degenerate circle given by (15). Set  $c^{-1} =: d + ie$  with  $d, e \in K$ . Then, by (1) and (2),

$$cKc^{-1} = \begin{cases} \{(cd)k + (cie)\bar{k} \mid k \in K\} : K/Z \text{ Galois,} \\ \{k + (ce)k^D \mid k \in K\} : K/Z \text{ not Galois.} \end{cases}$$

Now the assertion follows by Lemma 1. ■

### 3.7

Next we turn to non-degenerate circles.

**Theorem 3** *All non-degenerate circles of the affine plane  $\mathcal{A}_K(L)$  are in one orbit of  $\text{AGL}(1, L)$ .*

*Proof.* Let  $\mathcal{C}_0$  be the chain with transversal line

$$(1, 0)K \vee (i, i)K. \quad (16)$$

Then  $\Gamma_0 := \mathcal{C}_0^\rho$  is a non-degenerate circle with regular point 0.

Let  $K/Z$  be Galois. Then 1 is the other regular point of  $\Gamma_0$ . If  $\Gamma_1$  is a non-degenerate circle then there exists an affinity  $\alpha \in \text{AGL}(1, L)$  taking the regular points of  $\Gamma_1$  to 0 and 1, respectively. Hence  $\Gamma_1^{\alpha\rho^{-1}}$  is a chain with one transversal line through  $(1, 0)K$  and the other transversal line through  $(1, 1)K$ . Applying the collineation  $\iota$  on  $(1, 1)K$  establishes that (16) is a transversal line of this chain, whence  $\Gamma_0 = \Gamma_1^\alpha$ .

Now assume that  $K/Z$  is not Galois. If  $\Gamma_1$  is a non-degenerate circle then there exists an affinity  $\alpha \in \text{AGL}(1, L)$  taking the only regular point of  $\Gamma_1$  to 0. The chain  $\Gamma_1^{\alpha\rho^{-1}}$  has a unique transversal line through  $(1, 0)K$  and some point of the plane  $(i, 0)K \vee \infty$ , say

$$(id, e + if)K \quad \text{with } d, e, f \in K, \quad d, e + if \neq 0.$$

There exists an element  $m_1 \in L \setminus \{0\}$  such that  $m_1(e + if) = id$ . The collineation  $\kappa$  of  $\mathcal{P}_K$  given by  $(l_0, l_1)K \mapsto (l_0, m_1 l_1)K$  leaves  $\mathcal{S}_{L/K}$  invariant, fixes the point  $(1, 0)K$  as well as the line  $\infty$  and takes  $(id, e + if)K$  to  $(i, i)K$ . Hence the induced affinity  $\rho^{-1}\kappa\rho$  of  $\mathcal{A}_K(L)$  carries  $\Gamma_1^\alpha$  over to  $\Gamma_0$ . ■

### 3.8

The non-degenerate circle  $\Gamma_0$  arising from the chain  $\mathcal{C}_0$  with transversal line (16) has the parametric representation

$$\{ik_1(k_0 + ik_1)^{-1} \mid (0, 0) \neq (k_0, k_1) \in K^2\}; \quad (17)$$

cf. also [1, Satz 3.2]. Next we establish an equation for  $\Gamma_0$ :



**Theorem 4** *The non-degenerate circle  $\Gamma_0$  given by (17) equals the set of all points  $u + iv$  ( $u, v \in K$ ) satisfying<sup>8</sup>*

$$u = N(u + iv). \quad (18)$$

*Proof.* The term  $ik_1(k_0 + ik_1)^{-1}$  in formula (17) can be rewritten as follows: If  $K/Z$  is Galois then

$$\begin{aligned} ik_1(k_0 + ik_1)^{-1} &= ik_1(\overline{k_0} - ik_1) ((k_0 + ik_1)(\overline{k_0} - ik_1))^{-1} \\ &= (\mu_2 k_1 \overline{k_1} + i \overline{k_0} k_1)(k_0 \overline{k_0} + \mu_2 k_1 \overline{k_1})^{-1}, \end{aligned}$$

otherwise

$$\begin{aligned} ik_1(k_0 + ik_1)^{-1} &= ik_1(k_0 + k_1 + k_1 i) ((k_0 + ik_1)(k_0 + k_1 + k_1 i))^{-1} \\ &= (\mu_2 k_1^2 + ik_0 k_1) (k_0^2 + k_0 k_1 + (k_0 k_1)^D + \mu_2 k_1^2)^{-1}. \end{aligned}$$

Now, since

$$N(u + iv) = \begin{cases} u\overline{u} + \mu_2 v\overline{v} & : K/Z \text{ Galois,} \\ u^2 + uv + (uv)^D + \mu_2 v^2 & : K/Z \text{ not Galois,} \end{cases}$$

it is easily seen that all points of  $\Gamma_0$  are satisfying equation (18).

Conversely, let  $q + ir$  ( $q, r \in K$ ) be a solution of (18). If  $q = 0$  then  $r = 0$ , whence we have a point of  $\Gamma_0$ . Otherwise set

$$k_0 := \begin{cases} \mu_2 r \overline{q}^{-1} & : K/Z \text{ Galois,} \\ \mu_2 r q^{-1} & : K/Z \text{ not Galois,} \end{cases} \text{ and } k_1 := 1.$$

The point of  $\Gamma_0$  with these parameters equals  $q + ir$ . ■

### 3.9

We are able to say a little bit more about non-degenerate circles provided that  $K/Z$  is Galois. Formula (18) becomes

$$N(u + iv) - u = (u - 1 + iv) * (u + iv) = 0. \quad (19)$$

Thus, if we intersect each line through 0 with its orthogonal line through 1 then the set of all such points of intersection equals  $\Gamma_0$ . This is a nice analogon to a well-known property of opposite points on a Euclidean circle<sup>9</sup>.

**Theorem 5** *Let  $K/Z$  be Galois. Write  $E := \{y \in K \mid y + \overline{y} = 1\}$  and  $\mathcal{H}_e$  ( $e \in E$ ) for the affine Hermitian variety formed by all points  $u + iv$  ( $u, v \in K$ ) subject to the equation*

$$N(u + iv) = eu + \overline{eu}.$$

*Then the non-degenerate circle  $\Gamma_0$  given by (17) can be written as*

$$\Gamma_0 = \mathcal{H}_e \cap \mathcal{H}_f \quad \text{for all } e, f \in E \text{ with } e \neq f. \quad (20)$$

<sup>8</sup>In the elementary plane of complex numbers the same kind of equation gives a circle through 0 and 1.

<sup>9</sup>The points 0 and 1 are, however, the only points of  $\Gamma_0$  with this property.

*Proof.* A straightforward calculation yields

$$E = \begin{cases} \frac{1}{2} + (\lambda_1 + 2a)Z : \text{Char}K \neq 2, \\ a\lambda_1^{-1} + Z : \text{Char}K = 2, \end{cases}$$

whence  $E$  is infinite. Given  $q + ir \in \Gamma_0$  ( $q, r \in K$ ) then  $q \in Z$  implies

$$\Gamma_0 \subset \bigcap_{e \in E} \mathcal{H}_e.$$

Choose distinct elements  $e, f \in E$  and  $q + ir \in \mathcal{H}_e \cap \mathcal{H}_f$  ( $q, r \in K$ ). Then

$$N(q + ir) - N(q + ir) = eq + \overline{eq} - fq - \overline{fq} = 0.$$

But

$$\frac{e - f}{\overline{f} - \overline{e}} = 1,$$

so that  $q = \overline{q}$  and therefore  $q + ir \in \Gamma_0$ . ■

### 3.10

There is an alternative approach to  $\mathcal{A}_K(L)$  via the point model of  $\Sigma(K, L)$  on the Klein quadric  $\mathcal{Q}$ .

Write  $I := \infty^\gamma$  and  $\mathcal{Z}$  for the  $\gamma$ -image of the ruled plane on  $\tilde{\mathcal{A}}$ ; this  $\mathcal{Z}$  is a plane on the Klein quadric. Furthermore let  $\tilde{\mathcal{F}}$  be any plane of  $\hat{\mathcal{P}}_K$  skew to  $\mathcal{Z}$  and write

$$\pi : \hat{\mathcal{P}}_K \setminus \mathcal{Z} \rightarrow \tilde{\mathcal{F}} \quad (21)$$

for the projection with centre  $\mathcal{Z}$  onto the plane  $\tilde{\mathcal{F}}$ . It is well known from descriptive line geometry that there exists a collineation  $\psi$  of  $\tilde{\mathcal{A}}$  onto  $\tilde{\mathcal{F}}$  such that

$$(p \cap \tilde{\mathcal{A}})^\psi = p^{\gamma\pi}$$

for all lines  $p$  of  $\mathcal{P}_K$  not contained in  $\tilde{\mathcal{A}}$ . Cf., e.g., [3]. We turn  $\tilde{\mathcal{F}}$  into an affine plane  $\mathcal{F}$ , say, by regarding  $\tilde{\mathcal{F}} \cap I^\perp$  as its line at infinity; here  $I^\perp$  denotes the tangent hyperplane of the Klein quadric at  $I$ . Then  $\infty^\psi = \mathcal{F} \cap I^\perp$ .

The bijectivity of  $\rho$  implies that  $\mathcal{S}_{L/K}^\gamma \setminus \{I\}$  is mapped bijectively under  $\pi$  onto the affine plane  $\mathcal{F}$ . The restriction

$$\pi \mid \mathcal{S}_{L/K}^\gamma \setminus \{I\}$$

can be seen as a **generalized stereographic projection** of the oval quadric  $\mathcal{S}_{L/K}^\gamma$  of  $\Pi_Z$  onto the affine plane<sup>10</sup>  $\mathcal{F}$ .

Let  $\mathcal{C}$  be a chain. Then  $\mathcal{C}^\gamma = \mathcal{X} \cap \mathcal{Q} \cap \Pi_Z$  for some 3-dimensional subspace  $\mathcal{X}$  of  $\hat{\mathcal{P}}_K$ . We leave it to the reader to show that  $(\mathcal{C} \setminus \{\infty\})^{\gamma\pi}$  is an affine line if  $\mathcal{X} \cap \mathcal{Z}$  is a line through  $I$ , a degenerate circle if  $\mathcal{X} \cap \mathcal{Z} = \{I\}$  and a non-degenerate circle if  $\mathcal{X} \cap \mathcal{Z}$  is some point other than  $I$ .

Using the mapping  $\gamma\pi\psi^{-1}$  instead of  $\rho$  is very convenient to establish results on the images of traces [1, p.327], since their  $\gamma$ -images are just the regular conics on  $\mathcal{S}_{L/K}^\gamma$  [8, 3.4]. We sketch just one result without proof:

<sup>10</sup>A ‘usual’ stereographic projection would map onto a 4-dimensional affine space over  $Z$  rather than an affine plane over  $K$ .

Let  $\mathcal{C}$  be a chain through  $\infty$  such that  $(\mathcal{C} \setminus \{\infty\})^\rho =: \Delta$  is a degenerate circle of  $\mathcal{A}_K(L)$ . Then the  $\rho$ -images of traces in  $\mathcal{C}$  are on one hand the lines of the affine plane  $\Delta$  and on the other hand certain ellipses of  $\Delta$ . If these ellipses are extended to conics of  $\mathcal{A}_K(L)$  then the absolute directions of  $\Delta$  determine their points at infinity<sup>11</sup>. This is the well-known concept of absolute circular points.  $\Delta$  is a Euclidean plane representing the extension  $K/Z$ . Cf. [14].

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<sup>11</sup>There is only one such point if  $K/Z$  is not Galois.

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